Quotients of CI-groups are CI-groups

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Abstract

We show that a quotient group of a CI-group with respect to (di)graphs is a CI-group with respect to (di)graphs.

In [1, 2], Babai and Frankl provided strong constraints on which finite groups could be CI-groups with respect to graphs. As a tool in this program, they proved [1, Lemma 3.5] that a quotient group G/N of a CI-group G with respect to graphs is a CI-group with respect to graphs provided that N is characteristic in G. They were not able to prove that a quotient group of a CI-group with respect to graphs is a CI-group with respect to graphs in the general case, and so introduced the notion of a *weak CI-group with respect to graphs* in order to treat quotient groups of CI-groups. In some sense, the program that Babai and Frankl started was completed by Li [4] when he showed that all CI-groups are solvable. (Babai and Frankl mention in [2] a sequel to their first paper that addressed showing all CI-groups with respect to graphs are solvable. This sequel never appeared.) We will show that a quotient group of a CI-group with respect to (di)graphs is a CI-group with respect to (di)graphs. This will allow for a simplification of the proofs of Babai and Frankl in [1,2] (for example the notion of a weak CI-group with respect to graphs will no longer be needed), and consequently, as Li's proof in [4] was based on the earlier work of Babai and Frankl, a simplification of the proof that a CI-group with respect to graphs is solvable. We begin with some basic definitions.

Definition 1 Let G be a group and $S \subset G$. Define a **Cayley digraph of** G, denoted $\mathrm{Cay}(G, S)$,

to be the digraph with $V(\mathrm{Cay}(G,S)) = G$ and $E(\mathrm{Cay}(G,S)) = \{(g,gs) : g \in G, s \in S\}$. We call S the **connection set of** $\mathrm{Cay}(G,S)$. If $S = S^{-1}$, then $\mathrm{Cay}(G,S)$ is a graph.

Typically, definitions of Cayley (di)graphs assume $1_G \notin S$ to avoid loops, but this assumption is rarely material to proofs, and will not be made here.

It is straightforward to show that $g_L : G \to G$ by $g_L(x) = gx$ is always an automorphism of $\mathrm{Cay}(G,S)$, and so $G_L = \{g_L : g \in G\}$ is a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G,S))$, the automorphism group of $\mathrm{Cay}(G,S)$. G_L is the **left regular representation of** G.

Definition 2 We say that a group G is a **CI-group with respect to (di)graphs** if given $\mathrm{Cay}(G,S)$ and $\mathrm{Cay}(G,S')$, $S,S' \subset G$, then $\mathrm{Cay}(G,S)$ and $\mathrm{Cay}(G,S')$ are isomorphic if and only if $\alpha(S) = S'$ for some $\alpha \in \mathrm{Aut}(G)$.

It is also straightforward to verify that $\alpha(\mathrm{Cay}(G,S)) = \mathrm{Cay}(G,\alpha(S))$ is a Cayley (di)graph of G for every $S \subset G$ and $\alpha \in \mathrm{Aut}(G)$. Thus if one is testing whether or not two Cayley (di)graphs of a group G are isomorphic, one must always check whether or not there is a group automorphism of G that acts as an isomorphism. A CI-group with respect to (di)graphs is then a group where the group automorphisms of G are the only maps which need to be checked to determine isomorphism.

We now state some of the definitions from permutation group theory that will be required.

Definition 3 Let G be a transitive group acting on a set X. A subset $B \subseteq X$ is a **block** of G if whenever $g \in G$, then $g(B) \cap B \in \{\emptyset, B\}$. If $B = \{x\}$ for some $x \in X$ or $B = X$, then B is a **trivial block**. Any other block is nontrivial, and if G admits nontrivial blocks then G is **imprimitive**. If G is not imprimitive, we say that G is **primitive**. Note that if B is a block of G, then $g(B)$ is also a block of B for every $g \in G$, and is called a **conjugate block of** B. The set of all blocks conjugate to B, denoted $\mathcal{B}$, is a partition of X, and $\mathcal{B}$ is called a **G-invariant partition of** X.

Definition 4 Let $\mathcal{B}$ be a G-invariant partition. Define $\mathrm{fix}_G(\mathcal{B}) = \{g \in G : g(B) = B$ for all $B \in \mathcal{B}\}$. That is, $\mathrm{fix}_G(\mathcal{B})$ is the group of permutations in G that simultaneously fixes each block of $\mathcal{B}$ set-wise. If $\mathcal{C}$ is also a G-invariant partition and for every $C \in \mathcal{C}$ we have that $C \subset B$ for some $B \in \mathcal{B}$, we write $\mathcal{C} \preceq \mathcal{B}$. So $\mathcal{C}$ is a refinement of $\mathcal{B}$.

Wreath products of both groups and graphs will be crucial.

Definition 5 Let G be a permutation group acting on X and H a permutation group acting on Y. Define the **wreath product of G and H**, denoted $G \wr H$, to be the set of all permutations f of $X \times Y$ for which there exists $g \in G$, and for every $x \in X$ there exists $h_x \in H$, such that $f((x,y)) = (g(x), h_x(y))$.

We remark that many authors reverse the order of G and H in $G \wr H$, and/or refer to the wreath product of graphs (see Definition 8 below) as the lexicographic product.

The following result is certainly known by many readers. It and its proof are included here for completeness.

Lemma 6 *Let G and H be transitive groups and $\mathcal{B}$ the $(G \wr H)$-invariant partition formed by the orbits of $1_G \wr H$. If $\mathcal{C}$ is a $(G \wr H)$-invariant partition, then either $\mathcal{B} \preceq \mathcal{C}$ or $\mathcal{C} \preceq \mathcal{B}$. Consequently, $\mathcal{B}$ is the only $(G \wr H)$-invariant partition with blocks whose length is the degree of H.*

PROOF. Let $\mathcal{C}$ be a $(G \wr H)$-invariant partition, and $B \in \mathcal{B}$. Let K be the point-wise stabilizer of every point *not* in B. Then K is transitive on B. Now, either $\mathcal{B} \preceq \mathcal{C}$ or not. If so, we are finished. If not, then let $C \in \mathcal{C}$ such that $C \cap B \neq \emptyset$. Then there exists at least one element of B not in C, and so there exists $k \in K$ such that $k(C) \neq C$. Then $k(C) \cap C = \emptyset$ so that k fixes no point of C. But k fixes every point not in B, and so $C \subseteq B$ and $\mathcal{C} \preceq \mathcal{B}$. $\square$

Definition 7 Let Γ_1 and Γ_2 be digraphs. The **wreath product of Γ_1 and Γ_2**, denoted $\Gamma_1 \wr \Gamma_2$ is the digraph with vertex set $V(\Gamma_1) \times V(\Gamma_2)$ and edge set

$$\{(u,v)(u,v') : u \in V(\Gamma_1) \text{ and } vv' \in E(\Gamma_2)\} \cup \{(u,v)(u',v') : uu' \in E(\Gamma_1) \text{ and } v,v' \in V(\Gamma_2)\}.$$

The following result [3, Theorem 5.7] giving the automorphism group of vertex-transitive wreath product (di)graphs will be useful. In the statement, for a (di)graph Γ, $\bar{\Gamma}$ denotes the complement of Γ.

Theorem 8 *For any finite vertex-transitive (di)graph $\Gamma \cong \Gamma_1 \wr \Gamma_2$, if $\mathrm{Aut}(\Gamma) \neq \mathrm{Aut}(\Gamma_1) \wr \mathrm{Aut}(\Gamma_2)$ then there are some natural numbers $r > 1$ and $s > 1$ and vertex-transitive (di)graphs Γ_1' and Γ_2' for which either*

1. $\Gamma_1 \cong \Gamma_1' \wr K_r$, $\Gamma_2 \cong K_s \wr \Gamma_2'$ *or*
2. $\Gamma_1 \cong \Gamma_1' \wr \bar{K}_r$ *and* $\Gamma_2 \cong \bar{K}_s \wr \Gamma_2'$,

and $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Gamma_1') \wr (\mathcal{S}_{rs} \wr \mathrm{Aut}(\Gamma_2'))$.

Theorem 9 *Let G be a CI-group with respect to (di)graphs and $H \triangleleft G$. Then G/H is a CI-group with respect to (di)graphs.*

Proof. Let $\ell = |H|$, and $\mathrm{Cay}(G/H, S_1)$ and $\mathrm{Cay}(G/H, S_2)$ be isomorphic. If $\mathrm{Cay}(G/H, S_1) \neq \Gamma_1 \wr K_\ell$ for some (di)graph Γ_1 and $\ell \geq 2$, then $\mathrm{Cay}(G/H, S_2) \neq \Gamma_2 \wr K_\ell$ for any (di)graph Γ_2 and $\ell \geq 2$. In this case, define $T_1 = \{gh : gH \in S_1, h \in H\} \cup (H - \{1_G\})$ and $T_2 = \{gh : gH \in S_2, h \in H\} \cup (H - \{1_G\})$. Then $\mathrm{Cay}(G, T_1) = \mathrm{Cay}(G/H, S_1) \wr K_\ell$ and $\mathrm{Cay}(G, T_2) = \mathrm{Cay}(G/H, S_2) \wr K_\ell$ are isomorphic Cayley (di)graphs of G. Additionally, by Theorem 8, we have that $\mathrm{Aut}(\mathrm{Cay}(G, T_1)) = \mathrm{Aut}(\mathrm{Cay}(G/H, S_1)) \wr \mathcal{S}_\ell$ and $\mathrm{Aut}(\mathrm{Cay}(G, T_2)) = \mathrm{Aut}(\mathrm{Cay}(G/H, S_2)) \wr \mathcal{S}_\ell$. On the other hand, if $\mathrm{Cay}(G/H, S_1) = \Gamma_1 \wr K_\ell$ for some Γ_1 and $\ell \geq 2$, then $\mathrm{Cay}(G/H, S_2) = \Gamma_2 \wr K_\ell$ for some Γ_2. In this case, define $T_1 = \{gh : gH \in S_1, h \in H\}$ and $T_2 = \{gh : gH \in S_2, h \in H\}$. Then $\mathrm{Cay}(G, T_1) = \mathrm{Cay}(G/H, S_1) \wr \bar{K}_\ell$ and $\mathrm{Cay}(G, T_2) = \mathrm{Cay}(G/H, S_2) \wr \bar{K}_\ell$ are isomorphic Cayley digraphs of G. As before, by Theorem 8, we have that $\mathrm{Aut}(G, T_1) = \mathrm{Aut}(\mathrm{Cay}(G/H, S_1)) \wr \mathcal{S}_\ell$ and $\mathrm{Aut}(\mathrm{Cay}(G, T_2)) = \mathrm{Aut}(\mathrm{Cay}(G/H, S_2)) \wr \mathcal{S}_\ell$. In either case, $\mathrm{Cay}(G, T_1)$ and $\mathrm{Cay}(G, T_2)$ are isomorphic Cayley digraphs of G such that $\mathrm{Aut}(\mathrm{Cay}(G, T_1)) = \mathrm{Aut}(\mathrm{Cay}(G/H, S_1)) \wr \mathcal{S}_\ell$ and $\mathrm{Aut}(\mathrm{Cay}(G, T_2)) = \mathrm{Aut}(\mathrm{Cay}(G/H, S_2)) \wr \mathcal{S}_\ell$.

As G is a CI-group with respect to (di)graphs, there exists $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(\mathrm{Cay}(G, T_1)) = \mathrm{Cay}(G, \alpha(T_1)) = \mathrm{Cay}(G, T_2)$. Since both $\mathrm{Cay}(G, T_1)$ and $\mathrm{Cay}(G, T_2)$ have the form $\Gamma_1' \wr \Gamma_2'$ where Γ_2' has order ℓ, Lemma 6 tells us that there is a unique $\mathrm{Aut}(\mathrm{Cay}(G, T_1))$-invariant partition with blocks of length ℓ in $\mathrm{Cay}(G, T_1)$, and a unique $\mathrm{Aut}(\mathrm{Cay}(G, T_2))$-invariant partition with blocks of length ℓ in $\mathrm{Cay}(G, T_2)$, and furthermore that in each case, these block systems are formed by the orbits of $1_{\mathrm{Aut}(\mathrm{Cay}(G/H, S_i))} \wr S_\ell$. By inspecting the connection sets of $\mathrm{Cay}(G, T_1)$ and $\mathrm{Cay}(G, T_2)$, it is clear that in both graphs these orbits are the cosets of H in G. Since α is an isomorphism from $\mathrm{Cay}(G, T_1)$ to $\mathrm{Cay}(G, T_2)$, it must take the unique invariant partition with blocks of length ℓ in $\mathrm{Cay}(G, T_1)$, to the unique invariant partition with blocks of length ℓ in $\mathrm{Cay}(G, T_2)$, and hence take any coset of H to a coset of H. Since $\alpha \in \mathrm{Aut}(G)$ it takes subgroups of G to subgroups of G, so in particular, $\alpha(H) = H$.

Now α induces an automorphism $\bar{\alpha}$ of G/H defined by $\bar{\alpha}(gH) = \alpha(g)H$. Since $\alpha(H) = H$, this is well-defined. We claim that $\bar{\alpha}(\mathrm{Cay}(G/H, S_1)) = \mathrm{Cay}(G/H, \bar{\alpha}(S_1)) = \mathrm{Cay}(G/H, S_2)$, and so G/H is a CI-group with respect to digraphs. To see this, suppose that $gH \in S_1$. Then $\bar{\alpha}(gH) = \alpha(g)H$, and by the definition of T_1, $gh \in T_1$ for every $h \in H$. Since $\alpha(T_1) = T_2$, this means that $\alpha(gh) = \alpha(g)\alpha(h) \in T_2$ for every $h \in H$, and since $\alpha(H) = H$, this means $\alpha(g)h \in T_2$ for every $h \in H$. By definition of T_2, this means that $\bar{\alpha}(gH) = \alpha(g)H \in S_2$. Since gH was an arbitrary element of S_1, this shows that $\bar{\alpha}(S_1) = S_2$, as claimed. □

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